J. Fluid Mech. (1975), *vol.* 72, *part* 3, *pp.* 425–432
Printed in Great Britain

Instability of a horizontal layer of viscoelastic liquid on an oscillating plane

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(Received 16 August 1974 and in revised form 23 June 1975)

Secular instability of unsteady flow of a horizontal layer of viscoelastic liquid set in motion by simple harmonic motion of the lower boundary in its own plane is investigated. Using an extension of Floquet's theory for ordinary differential equations, the stability of long waves is studied by a regular perturbation method. The elastic parameter of the fluid is found to be destabilizing and stabilizing in different ranges of frequency.

1. Introduction

It has been recognized that problems of stability of unsteady basic flows present several difficulties in view of the fact that their time dependence precludes the use of an exponential time factor for the perturbation quantities. Benjamin & Ursell (1954) found that, when a cylinder containing inviscid liquid with a free surface is moved up and down with a simple harmonic acceleration of amplitude a_0, the liquid may be unstable even if a_0 is much smaller than the gravitational acceleration g. Donnelly, Reif & Suhl (1962) experimentally demonstrated that the onset of instability in the form of Taylor ring vortices in the flow between two concentric cylinders with the inner rotating and the outer at rest can be delayed when the angular velocity of inner cylinder is modulated about its mean value. While considering Kelvin–Helmholtz instability of an interface between two streams of fluid in relative tangential motion, Kelly (1964) found that unsteadiness in the basic flows gives rise to parametric amplification of the interface motion. Rosenblat (1968) investigated the stability of time-periodic inviscid azimuthal flows between coaxial circular cylinders to axisymmetric disturbances.

To the best of the authors' knowledge, the stability characteristics of an unsteady basic flow of a non-Newtonian liquid have not received adequate attention despite the fact that the literature abounds in studies of the stability of steady basic flows of non-Newtonian liquids. Recently Yih (1968) investigated the stability of a horizontal layer of viscous liquid with a free surface, the flow being caused by the oscillation of the lower rigid boundary in its own plane. The purpose of the present investigation is to extend this problem to the class of non-Newtonian (viscoelastic) liquids known as incompressible second-order fluids, and the stability characteristics of long waves in such flows are studied by a

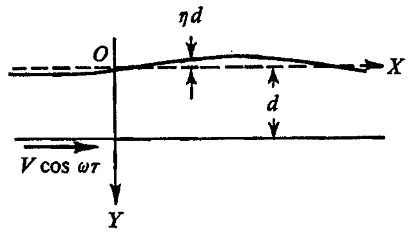

FIGURE 1. Sketch of the physical problem.

perturbation method due to Yih. In our investigation the primary flow is completely unsteady in the sense that it has no steady part whatsoever and we study the instability of this flow with respect to infinitesimal disturbances.

2. Mathematical formulation

A horizontal layer of a second-order fluid of depth d and density ρ is set into motion by movement of the lower rigid boundary with velocity $V \cos \Omega t$ in the X direction, Ω being the frequency and V the amplitude (figure 1). Using the postulate of gradually fading memory, Coleman & Noll (1960) derived the following constitutive equation for an incompressible second-order fluid (which is isotropic):

$$\tau_{ij} = -p\delta_{ij} + \eta_0 A_{(1)ij} + \beta_0 A_{(1)ik} A_{(1)kj} + \nu_0 A_{(2)ij}, \tag{1}$$

where $\boldsymbol{\tau}$ is the stress tensor, p is an indeterminate pressure and η_0, β_0 and ν_0 are material constants. The rate-of-strain tensor $\mathbf{A}_{(1)}$ and the acceleration tensor $\mathbf{A}_{(2)}$ are defined by

$$A_{(1)ij} = v_{i,j} + v_{j,i}, \tag{2}$$

$$A_{(2)ij} = a_{i,j} + a_{j,i} + 2v_{m,i} v_{m,j}, \tag{3}$$

where a_i's are the acceleration components, given by $\partial v_i/\partial t + v_j v_{i,j}$. It is important to point out that such a fluid exhibits the normal-stress effects which are generally observed in flows of dilute polymer solutions, e.g. polyethylene oxide in water (Polyox) or polyisobutylene in cetane. Further, (1) is valid for low shear rates and $\nu_0 < 0$ from thermodynamic considerations.

We introduce the following dimensionless variables:

$$x = X/d, \quad y = Y/d, \quad \tau = Vt/d, \tag{4}$$

t being the time. The equation governing the basic flow reduces, on using (1), to

$$\frac{\partial U}{\partial \tau} = \frac{1}{R}\frac{\partial^2 U}{\partial y^2} - M\frac{\partial^2}{\partial y^2}\left(\frac{\partial U}{\partial \tau}\right), \tag{5}$$

where

$$U = \bar{u}/V, \quad R = \rho V d/\eta_0, \quad M = -\nu_0/\rho d^2, \tag{6}$$

$\bar{u}(Y, t)$ being the velocity of the basic flow in the X direction. The boundary conditions are vanishing shear stress at the free surface and the no-slip condition at the rigid wall and are given by

$$\frac{\partial U}{\partial y} - MR\frac{\partial}{\partial \tau}\left(\frac{\partial U}{\partial y}\right) = 0 \quad \text{at} \quad y = 0, \tag{7}$$

$$U = \cos \omega\tau \quad \text{at} \quad y = 1, \tag{8}$$

with $\omega = \Omega d/V$. Assuming $\partial U/\partial y = 0$ at $\tau = 0$, equation (7) reduces to

$$\partial U/\partial y = 0 \quad \text{at} \quad y = 0 \quad \text{for} \quad \tau > 0. \tag{9}$$

The solution of (5) satisfying (8) and (9) is

$$U(y,\tau) = A_1[W + W^* - i \tanh \beta_1 \tan(\beta_1 S)(W - W^*)], \tag{10}$$

where an asterisk denotes a complex conjugate and

$$W = 2\cosh[\beta_1(1+iS)y]\, e^{i\omega\tau}, \tag{11}$$

$$A_1 = \cosh\beta_1 \cos\beta_1 S/4[\cos^2\beta_1 S + \sinh^2\beta_1], \tag{12}$$

$$\beta_1 = \left[\frac{R\omega\{(1+M^2R^2\omega^2)^{\frac{1}{2}} - MR\omega\}}{2(1+M^2R^2\omega^2)}\right]^{\frac{1}{2}}, \tag{13}$$

$$S = (1+M^2R^2\omega^2)^{\frac{1}{2}} + MR\omega. \tag{14}$$

Notice that U as given by (10) is real.

We now perturb the flow given by (10) and take as the dimensionless forms of the perturbed velocity components and pressure

$$u = U + u', \quad v = v', \quad p = P + p', \tag{15}$$

using as units of velocity and pressure V and ρV^2 respectively. Introducing the stream function $\psi(x,y,\tau)$ such that $u' = \partial\psi/\partial y$ and $v' = -\partial\psi/\partial x$ and following Yih (1968) exactly but incorporating the terms due to viscoelasticity, the modified Orr–Sommerfeld equation is

$$R[(\partial/\partial\tau + i\alpha U)\{\phi_{yy} - \alpha^2\phi + M(\phi_{yyyy} - 2\alpha^2\phi_{yy} + \alpha^4\phi)\} - iM\alpha U_{yyyy}\phi - i\alpha U_{yy}\phi] = \alpha^4\phi - 2\alpha^2\phi_{yy} + \phi_{yyyy}, \tag{16}$$

where a subscript denotes differentiation and

$$\psi = \phi(y,\tau)\, e^{i\alpha x}. \tag{17}$$

Similarly the boundary conditions on the tangential and normal stress at the free surface are

$$[U_{yy}(0,\tau) - MRU_{yy\tau}(0,\tau)]h(\tau) - MR[(\partial/\partial\tau + i\alpha U)(\phi_{yy} + \alpha^2\phi)]_{y=0} + [\phi_{yy} + \alpha^2\phi + i\alpha MRU_{yy}\phi]_{y=0} = 0 \tag{18}$$

and

$$i\alpha[F^{-2} + S_1\alpha^2]h(0) + R^{-1}[\phi_{yyy} - 3\alpha^2\phi_y]_{y=0} - i\alpha M[U_{yy}\phi_y - U_{yyy}\phi]_{y=0} - [(\partial/\partial\tau + i\alpha U)\{\phi_y + M(\phi_{yyy} - 3\alpha^2\phi_y)\}]_{y=0} = 0. \tag{19}$$

In the above equations, F is the Froude number $V/(gd)^{\frac{1}{2}}$, $S_1 = T/\rho V^2 d$, T being the surface tension, and the deformation of the free surface η is taken as

$$\eta = h(\tau)\, e^{i\alpha x}. \tag{20}$$

The no-slip conditions at the lower boundary are

$$\phi(1,\tau) = 0, \quad \phi_y(1,\tau) = 0. \tag{21}$$

Further, the kinematic condition at the free surface is

$$[\partial/\partial\tau + i\alpha U(0,\tau)]\, h = -i\alpha\phi(0,\tau). \tag{22}$$

Thus the stability characteristics of the flow are governed by (16) subject to the boundary conditions (18), (19), (21) and (22).

We next study the above differential system by an extension of the well-known Floquet theory for ordinary differential equations with periodic coefficients (Coddington & Levinson 1955, p. 78) to the realm of partial differential equations. In fact this linear stability analysis for long-wave disturbances follows exactly that in Yih (1968) with $\phi(y,\tau)$ and $h(\tau)$ taken as

$$\phi(y,\tau) = \exp(\mu_1\tau)\,[\phi_0(y,\tau)+\alpha\phi_1(y,\tau)+\alpha^2\phi_2(y,\tau)+\ldots], \tag{23}$$

$$h(\tau) = \exp(\mu_1\tau)\,[h_0(\tau)+\alpha h_1(\tau)+\alpha^2 h_2(\tau)+\ldots], \tag{24}$$

where the functions ϕ_j and h_j are periodic in τ and the Floquet exponent μ_1 has the form

$$\mu_1 = \theta_0+\alpha\theta_1+\alpha^2\theta_2+\ldots. \tag{25}$$

Omitting the details of the analysis, we eventually find that $\theta_0 = \theta_1 = 0$ and the flow is unstable or stable according as θ_2 is positive or negative. This criterion turns out to be

$$\frac{48A_1^3(1+\nu'^2)(\nu'+MR\omega)MR\omega}{1+M^2R^2\omega^2}+\frac{3A_1^2(1+\nu'^2)}{\omega R}(R_2'-R_1') \gtrless F^{-2}, \tag{26}$$

where
$$\nu' = \tanh\beta_1 . \tan\beta_1 S \tag{27}$$

and R_1' and R_2' are respectively the real parts of

$$\frac{2(1-M^2R^2\omega^2)i}{S^2}[(S^4\sinh 2\beta_1+i\sin 2\beta_1 S)\tanh\{\beta_1(1+iS)\}]+\frac{8i\cosh[\beta_1(1+iS)]}{\cosh[\beta_1(1-iS)]} \tag{28}$$

and
$$\frac{4(1-M^2R^2\omega^2)\beta_1 i}{S}[(S^3\cosh 2\beta_1+i\cos 2\beta_1 S)\tanh\{\beta_1(1+iS)\}] + \frac{8i\beta_1(1+iS)\sinh[\beta_1(1+iS)]}{\cosh[\beta_1(1-iS)]}. \tag{29}$$

In the notation of Yih (1968), $A = 2A_1$, $\nu = \nu'$, $R_1 = \frac{1}{8}R_1'$ and $R_2 = \frac{1}{8}R_2'$ when $M = 0$. In this case, (26) gives the criterion for instability or stability as

$$\frac{6A^2(1+\nu^2)}{\omega R}(R_2-R_1) \gtrless F^{-2},$$

which agrees with Yih (1968).

In the stability criterion (26), the left-hand side depends on ωR and ωRM only, and these parameters do not involve the velocity scale V. Therefore for an oblique wave, with V effectively reduced to $V\cos\theta$, only the right-hand side of (26) is altered, from F^{-2} to $(F\cos\theta)^{-2}$. This shows that two-dimensional disturbances will be more unstable than the three-dimensional ones, i.e. Squire's theorem holds. Of course, the fact that the left-hand side of (26) is independent of V would have been more apparent had the proper Reynolds number scaling been used, e.g. V in (6) replaced by $V^* = \Omega d$.

It may be noted that the neglect of the surface-tension term $S_1\alpha^2$ compared

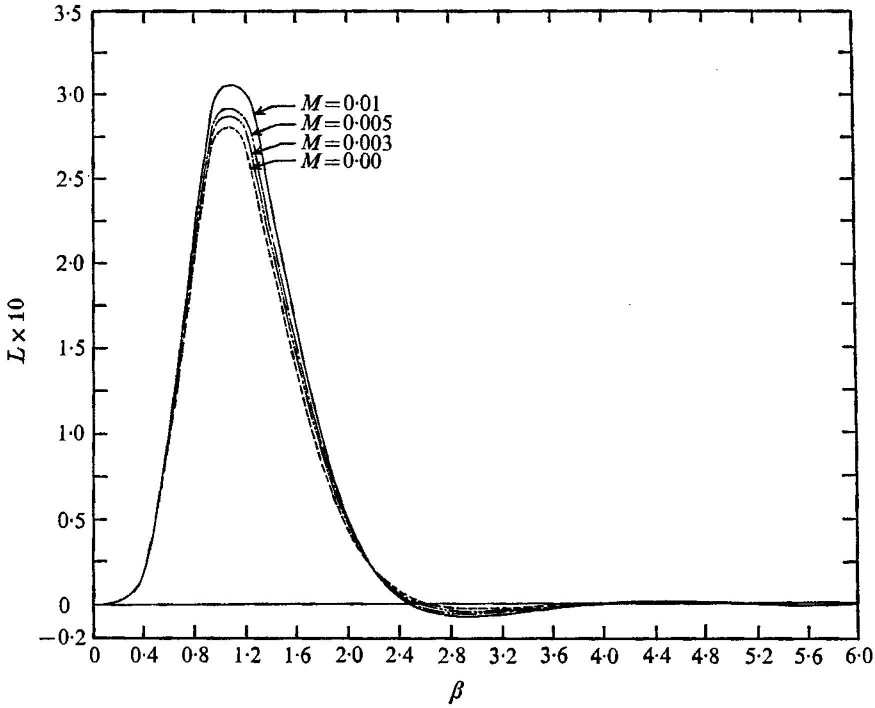

FIGURE 2. Variation of L with β for several values of M.

with F^{-2} in (19) (as we have done in our analysis), although formally correct for α sufficiently small and F and T fixed, may be rather restrictive. For

$$S_1\alpha^2/F^{-2} = (T/\rho g)(4\pi/\lambda)^2,$$

where λ is the physical wavelength, and this is small only for $\lambda \gg 10$ cm. However this restriction may be relaxed on replacing F^{-2} by $F^{-2}+S_1\alpha^2$ in (26).

Figure 2 shows the variation of L, the left-hand side of (26), with respect to $\beta = (\frac{1}{2}\omega R)^{\frac{1}{2}}$ for different values of the elastic parameter M. The calculations have been performed on an IBM 1620 Digital Computer. Each curve has a maximum such that when F^{-2} exceeds this maximum the flow is stable. As M increases, the maximum of each curve also increases, and this establishes the destabilizing role of the elastic properties of the liquid for a certain frequency range. Equation (26) shows that, when $L > 0$, there can be instability of the flow even for small Reynolds numbers R if F is sufficiently large. It may be noticed from figure 2 that, for certain ranges of the frequency parameter β, L becomes negative, which means that in this range the oscillation of the plate stabilizes the flow against long-wave disturbances. It is of some interest to note that this frequency range of stabilized motion increases with increasing M. This non-monotonic behaviour of the elastic parameter in the different frequency ranges contrasts sharply with the corresponding behaviour when the basic flow is steady. In fact, the visco-elastic terms in a second-order fluid were found to be destabilizing in the case of a steady basic flow of a liquid layer down an inclined plane (Gupta 1967). Of course these results are valid only when the elastic parameter is sufficiently small (say, $M \sim 10^{-3}$) and the frequency of the disturbance is not too large. In the present problem we have also taken $M \sim 10^{-3}$, so that the second-order fluid model can be shown to be a consistent constitutive approximation (Denn, Sun & Rushton 1971). It may be noted that in figure 2 we have also included the case $M = 0{\cdot}01$, for which $MR\omega \sim 0{\cdot}03$ at $\beta_1 = 1{\cdot}2$. For larger β values, $MR\omega$ clearly becomes large and invalidates the use of the second-order fluid model.

3. Discussion

We now make several comments regarding the validity of the model of the second-order fluid used in our analysis. This model may give rise to spurious instability in the form of instability of the rest state (Gupta 1967; Platten & Schechter 1970) if the approximations involved in deriving (1) are treated as if they are exact. In fact, although Gupta's (1967) stability analysis of low-frequency surface waves in a layer of a second-order fluid flowing down an inclined plane is correct, his treatment of high-frequency shear waves by the same model is of doubtful validity, as pointed out by Craik (1968). This is due to the possible breakdown of the model for very high frequencies, and, using a constitutive relation which takes account of the entire strain history of the motion, Craik showed that such anomalous behaviour does not occur. He also pointed out that a second-order fluid should not be expected to yield significant results at large wavenumbers (equivalently, small disturbance times). If, however, the disturbance time scale is large compared with the characteristic time scale (relaxation time) of the fluid, then the second-order fluid model is an internally consistent approximation to the stress-relaxing fluid due to Oldroyd (1950). As was pointed out by Porteous & Denn (1972), this would happen if $M \ll 1$ and $RM \ll 1$. Since we have considered surface wave instability for sufficiently small M, these conditions are fulfilled in addition to $\alpha \ll 1$.

We conclude our discussion by pointing out the justification for the physical significance of the problem. In fact there are several considerations which provide motivation for this study.

(i) In view of the importance of viscoelastic liquids like dilute polymer solutions in chemical industries, particularly the part played by viscoelasticity in drag reduction, several theoretical and experimental investigations of flow of such liquids have been carried out. Many interesting features are revealed through study of the response of such liquids to unsteady inputs, e.g. the measurement of viscoelastic properties of polymer solutions subject to oscillatory stress or strain by Tanner & Simmons (1967) and by Dodge & Krieger (1971). But so far no attempt has been made to see whether such unsteady flows are dynamically possible or not. The present study is specifically addressed to this problem. In fact it is of some interest to see whether unsteadiness in the basic flow of such liquids leads to parametric amplification of disturbances.

(ii) Chan Man Fong & Walters (1965) investigated the linear stability of plane Poiseuille flow of a second-order fluid and found elasticity to be destabilizing. This destabilizing influence of viscoelasticity appears to be in agreement with the experimental results of Jones & Maddock (1966) and Barnes & Walters (1967), but not with those of Giles & Pettit (1967). Our study reveals the novel feature of viscoelasticity playing a stabilizing role in a certain frequency range in addition to its usual destabilizing influence. We feel that this study may throw some light on the apparent discrepancy between various experimental studies like those we have mentioned above.

The reason why elasticity in our analysis is stabilizing for a large frequency of oscillation of the plate may be traced to the fact that the basic unsteady flow is of the boundary-layer type for high frequencies (akin to Stokes layer) and there is strong experimental evidence to support a stabilizing influence of elasticity on boundary-layer flows (Merrill, Smith & Chung 1966). However the exact mechanism of this stabilizing influence of elasticity is still not very clear.

We dedicate this paper to Professor C.-S. Yih, who is an unfailing source of help and inspiration to us.

Appendix

A few remarks regarding the convergence of the regular perturbation expansion [equation (23)] for $\phi(y,\tau)$ (or $h(\tau)$) in powers of α are in order.

We start with an ordinary differential equation

$$\frac{d^n f}{dx^n}+p_{n-1}(\alpha,x)\frac{d^{n-1}f}{dx^{n-1}}+p_{n-2}(\alpha,x)\frac{d^{n-2}f}{dx^{n-2}}+\ldots+p_0(\alpha,x)f=0,$$

where the $p_j(\alpha,x)$ are polynomials or entire functions of α and x. A solution of this equation in powers of α can be proved to be convergent for all finite values of α. A simple example is $f'+\alpha f=0$, whose solution ($\exp(-\alpha x)$) in powers of α is convergent for all finite values of α. Similarly the solution of the Orr–Sommerfeld equation in powers of α (as in the long-wave analysis of Yih 1963) can be shown to be convergent for a smooth basic velocity profile. The extension of this to partial differential equations is direct. The coefficients of the various partial derivatives of ϕ in the equation (16), governing $\phi(y,\tau)$, are also clearly entire functions of α (and y), the coefficient of the highest-order derivative being a constant, independent of α. Thus we may expect the series expansion in α to be convergent at any rate for small α.

However, a rigorous proof of the convergence of the expansion (23) could perhaps be given by the method of majorants. If it is possible to show that the ϕ's in (23) are uniformly bounded in the interval $0 \leqslant y \leqslant 1$, such that $|\phi_j(y,\tau)| \leqslant B$ for all j (B being a fixed constant), then the series in (23) is majorized by the series $B(1+\alpha+\alpha^2+\ldots)$, which will converge when $\alpha < 1$. This establishes the absolute and uniform convergence of the series (23). But since the ϕ_j's become increasingly complex with increasing j and no general pattern for the ϕ_j emerges from the analysis, the proof of uniform boundedness of the ϕ_j presents insuperable difficulties. We therefore leave the rigorous proof of convergence of (23) to more capable minds. All that we can say is that the series (23) is likely to be convergent for sufficiently small α, an assumption which is consistent with our long-wave analysis.

REFERENCES

BARNES, H. A. & WALTERS, K. 1967 *Nature*, **216**, 366.

BENJAMIN, T. B. & URSELL, F. 1954 *Proc. Roy. Soc.* A **225**, 505.

CHAN MAN FONG, C. F. & WALTERS, K. 1965 *J. Mécanique*, **4**, 439.

CODDINGTON, E. A. & LEVINSON, N. 1955 *Theory of Ordinary Differential Equations*. McGraw-Hill.

COLEMAN, B. D. & NOLL, W. 1960 *Arch. Rat. Mech. Anal.* **6**, 355.

CRAIK, A. D. D. 1968 *J. Fluid Mech.* **33**, 33.

DENN, M. M., SUN, Z. S. & RUSHTON, B. D. 1971 *Trans. Soc. Rheol.* **15**, 415.

DODGE, J. S. & KRIEGER, I. M. 1971 *Trans. Soc. Rheol.* **15**, 589.

DONNELLY, R. J., REIF, F. & SUHL, H. 1962 *Phys. Rev. Lett.* **9**, 353.

GILES, W. B. & PETTIT, W. T. 1967 *Nature*, **216**, 470.

GUPTA, A. S. 1967 *J. Fluid Mech.* **28**, 17.

JONES, W. M. & MADDOCK, J. L. 1966 *Nature*, **212**, 388.

KELLY, R. E. 1964 *M.I.T. Fluid Dyn. Res. Lab. Rep.* no. 64-3 (AFOSR 64-1827).

MERRILL, E. W., SMITH, K. A. & CHUNG, R. Y. C. 1966 *A.I.Ch.E. J.* **12**, 809.

OLDROYD, J. G. 1950 *Proc. Roy. Soc.* A **200**, 523.

PLATTEN, J. & SCHECHTER, R. S. 1970 *Phys. Fluids*, **13**, 832.

PORTEOUS, K. C. & DENN, M. M. 1972 *Trans. Soc. Rheol.* **16**, 295.

ROSENBLAT, S. 1968 *J. Fluid Mech.* **33**, 321.

TANNER, R. I. & SIMMONS, J. M. 1967 *Chem. Engng Sci.* **22**, 1803.

YIH, C.-S. 1963 *Phys. Fluids*, **6**, 321.

YIH, C.-S. 1968 *J. Fluid Mech.* **31**, 737.